ABSTRACT

for the degree Dissertation on the topic "Modeling of underground geophysical fields using by Helmholtz potential methods" for the degree Doctor of Philosophy (PhD) in the specialty "6D070500 – Mathematical and computer modeling" by Les Aidana

Relevance of the research topic. In modern mathematical physics, especially in quantum mechanics, acoustics, and electrodynamics, the Helmholtz equation is one of the fundamental models. This equation describes stationary wave processes, and studying the properties of its solutions is of great theoretical and practical importance. The existence and uniqueness of solutions to the Helmholtz equation were first considered in the works of Sommerfeld, where the radiation condition was introduced to describe solutions in unbounded domains. However, under classical Dirichlet conditions, uniqueness is violated, which requires the introduction of additional conditions. In this context, replacing the Sommerfeld radiation condition with an equivalent boundary condition in a bounded domain is an important scientific problem. Solving this problem enables the effective use of numerical modeling and computational methods.

Purpose and objectives of the study: – to study fundamental solutions of the Helmholtz equation in space;
– to replace Sommerfeld radiation conditions with integral boundary conditions in bounded domains;
– to prove properties of the Helmholtz potential under conditions;
– to formulate the Sommerfeld problem correctly in multidimensional space;
– to determine transparent boundary conditions in infinite cylindrical domains;
– to find conditions for determining the density of elliptic potentials;
– to develop and implement numerical algorithms for two-dimensional problems.

Scientific novelty of the research: – a fundamental solution of the Helmholtz equation in space is obtained and its asymptotic behavior is described for even nnn;
– the equivalence of the Helmholtz potential to a differential equation and a special integral boundary condition is proven for any;
– the Sommerfeld problem is generalized to multidimensional space and its equivalence to the Helmholtz potential is shown;
– analytical forms of transparent boundary conditions and their asymptotics for infinite semi-cylindrical domains are obtained;
– necessary and sufficient conditions for determining the density of multidimensional elliptic potentials are derived;
– an efficient numerical method for the two-dimensional Helmholtz equation is proposed and implemented.

Research methods. The study uses methods of partial differential equations, functional analysis, integral operator theory, potential theory, generalized function

theory, and special function theory. Boundary integral equation methods and numerical analysis techniques were also applied.

Theoretical significance of the research. The obtained results make a significant contribution to the development of the theory of elliptic differential equations in multidimensional spaces. New equivalent formulations for the Helmholtz equation deepen the theoretical foundation and provide directions for further research. The work is mainly theoretical in nature. The results can be used for further development of the theory of differential equations in space. In addition, the results can be applied in natural sciences, especially in geophysics for solving various problems in wells.

Practical significance of the research. The research results can be applied in fields such as geophysics, acoustics, and electromagnetic wave propagation. In particular, they can be used for modeling wave fields in wells, subsurface exploration, and solving engineering problems.

Approbation of the work.

1. Kalmenov T.Sh., Les A.K. Definition of the density of the elliptic potential. In: International Conference. Actual Problems of Analysis, Differential Equations and Algebra. (EMJ-2019).
2. Kalmenov T.Sh., Les A.K. Spectral decomposition of the Helmholtz potential. Traditional international scientific April conference dedicated to the Day of Scientists, April 3-5, 2019.
3. 3. Kalmenov T.Sh., Les A.K. The Sommerfeld problem and the inverse problem for the Helmholtz equation. Traditional international April mathematical conference in honor of the Day of Scientists of the Republic of Kazakhstan, dedicated to the 1150th anniversary of Abu Nasyr al-Farabi and the 75th anniversary of the Institute of Mathematics and Mathematical Modeling (Almaty - 2020).
4. 4. Iskakova U., Les A.K. On an overdetermined problem for the Lavrentiev-Bitsadze equation. International scientific conference dedicated to the 90th anniversary of the birth of Academician T.D. Dzhuraev (October 24–26, 2024, Tashkent, Uzbekistan). – Tashkent. Publishing house "Ma'rifat". 2024.166 p.
5. Kal'menov T.Sh., Les A.K. The Sommerfeld problem in an infinite semicylinder. Traditional International April Mathematical Conference in Honor of Science Day and the 80th Anniversary of the Institute of Mathematics and Mathematical Modeling, April 1–4, 2025.

Publications.

1. T.S. Kalmenov , S.I. Kabanikhin and Aidana Les. The Sommerfeld problem and inverse problem for the Helmholtz equation.Journal Inverse Ill-Posed Probl, 2021; 29(1): 49–64. h-index=2. Web of Science Impact factor=1.1 (Q1)

2. T.S. Kalmenov, A.K. Les and U. A. Iskakova. Determination of density of elliptic potential.Eurasian mathematical journal, ISSN 2077-9879, Vol. 12, N. 4 (2021), (Q3) p.43-52.

The list of articles proposed by the Committee for Quality Assurance in Science and Higher Education of the Ministry of Education and Science of the Republic of Kazakhstan includes the following publications:

T.S.Kalmenov and A.K.Les. Boundary condition of volume potential and its application to inverse problem.Kazakh Mathematical Journal, ISSN 2413–6468 19:4 (2019) p.71–78

Structure and volume of the dissertation. The dissertation is written in Kazakh and presented as a manuscript. It consists of a title page, contents, introduction, four main chapters, conclusion, and references. The total volume is 96 pages, including 29 figures and 1 appendix.

Main content of the dissertation. A dissertation containing the main content. Isaac Newton studied the physical meaning of potential as a law of universal gravitation. Gauss studied electromagnetic fields using this potential, as did scientists who further developed it—Lagrange, Laplace, Green, Sobolev, and others. Using Newton's potential, one can solve problems in mathematical physics—the Dirichlet problem, the Neumann problem, and many others. Newton's potential greatly influenced the creation of a new program in mathematics—the theory of general functions and the development of induction theorems. Currently, volume potentials are widely used to solve boundary value problems of second-order elliptic equations of theoretical and applied significance.

Chapter 1 In mathematical physics, the wave equation

$$\frac{\partial^2}{\partial t^2}\tilde{u} - a^2\Delta_x\tilde{u} = \tilde{f}(x,t) \quad (1)$$

The study uses its fundamental solution. Here $\Delta_x\tilde{u} = \sum_{i=1}^{3}\frac{\partial^2\tilde{u}}{\partial x_i^2}$ - Laplace operator. Equations (1) $\tilde{f}(x,t) = f(x,t) + u_0(x)\delta^{(1)}(t) + u_1(x)\delta(t)$the general solution of the Cauchy problem is written in the form of a cluster

$$\tilde{u}(x,t) = \left(\varepsilon * \tilde{f}\right)(x,t) \quad (2)$$

The solution of functions (2) here is given by Kirchhoff's formula$f \in C^2(t \geq 0)$, $u_0 \in C^3(R^3),\ u_1 \in C^2(R^3)$.

$$u(x,t) = \frac{1}{4\pi a^2}\int_{U(x,at)}\frac{f\left(\xi, t-\left|\frac{x-\xi}{a}\right|\right)}{|x-\xi|}d\xi$$

$$+\frac{1}{4\pi a^2 t}\int_{S(x,at)}\tilde{u}_1(\xi)dS+\frac{1}{4\pi a^2}\frac{\partial}{\partial t}\left[\frac{1}{t}\int_{S(x,at)}\tilde{u}_0(\xi)dS\right] \quad (3)$$

Studying the properties of time using formula (3) is very difficult. Therefore, in practice, the properties of soliton (independent wave) solutions are studied. If $\tilde{u}(x,t)=e^{ikt}u(x),\ \tilde{f}(x,t)=e^{ikt}f(x)$ We$|x|\to\infty$ we substitute , then $\tilde{u}(x,t)$equation (1) is reduced to the following Helmholtz equation

$$Lu=-\Delta_x u-k^2u=f(x),\ \ x\in R^3 \quad (4)$$

Here k is any real number, if $k=0$, then equation (4) becomes the Laplace equation.

$$Lu=-\Delta_x u=f(x),\ \ x\in R^3$$

A fundamental $\delta(x)$solution to the Helmholtz equation is constructed, obtained by deriving the fundamental solution of the multidimensional Helmholtz equation and the Macdonald function.

Chapter 2 The equivalence of the Helmholtz potential to the boundary value for any number k is proved.

Chapter 3 R n is simultaneously odd-dimensional (n = 2s + 3) and even-dimensional (n = 2s). the uniqueness of the solution of the Sommerfeld problem in space and Sommerfeld in the case of radiation with a finite base potential in a finite region of radiation conditions The German physicist Sommerfeld was the first to study the problems of quantum mechanics and the propagation of electromagnetic waves in space, proving the equivalence of the boundary condition.

I - Sommerfeld. R^3into the Helmholtz equation

$$-\Delta_x u-k^2u=f(x),\ \ x\in R^3$$

find a solution that satisfies these conditions at infinity

$$\lim_{|x|\to\infty}|u(x)|_{|x|=d}\le\frac{M}{d},\quad \lim_{|x|\to\infty}\left|\frac{\partial u}{\partial|x|}-iku\right|_{|x|=d}\le\frac{M}{d^2}$$

II - Sommerfeld. In $R^3$the Helmholtz equation

$$-\Delta_x u-k^2u=f(x),\ \ x\in R^3$$

find a solution that satisfies these conditions at infinity

$$\lim_{|x|\to\infty}|u(x)|_{|x|=d}\le\frac{M}{d},\quad \lim_{|x|\to\infty}\left|\frac{\partial u}{\partial|x|}+iku\right|_{|x|=d}\le\frac{M}{d^2} \quad (3)$$

In his work, Sommerfeld showed that if $suppf \subset \Omega$,Ω If the domain is bounded, then Sommerfeld's Problem I and Problem II have unique solutions. Sommerfeld's result raises the following question: Ω what condition can be used to replace Sommerfeld's Problem I in a bounded domain, that is, how Ωcan Sommerfeld's reflection condition be replaced by a boundary condition in a bounded domain? If such a condition is found, then a numerical solution to Sommerfeld's problem can easily be found. In this chapter, it is proved that this condition is a boundary condition for the Helmholtz potential.

Chapter 4 The main goal of the fourth section is to present numerical experiments on the Helmholtz problem and the Helmholtz potential in the specified domain for the two-dimensional case. This means, firstly, demonstrating the correspondence between the boundary value problem and its analytical solution, and secondly, demonstrating using numerical methods that the density of the inverse problem posed for the Helmholtz potential is unique. First, we consider one-dimensional analogs of the problems. When solving nonlocal boundary value problems numerically, the error of the approximate solution is significantly higher than that of local problems. Therefore, using an exact solution for nonlocal problems yields more efficient and accurate results. Since the proposed method allows for an exact expression of the solution, approximation errors arising during the numerical implementation of nonlocal problems are completely eliminated.

In this regard, this method is an effective tool, ensuring high accuracy and stability in solving nonlocal boundary value problems. There is a boundary condition for the potential for the Helmholtz equation. The solution to the problem is obtained on the basis of numerical results .

In conclusion. The fundamental solution of the Helmholtz equation is constructed in multidimensional spaces using Macdonald functions. The equivalence of the Helmholtz potential to the boundary value problem is proven for any. The uniqueness of the Sommerfeld problem solution in multidimensional space and the equivalence of the radiation condition to a potential boundary condition in a bounded domain are established. The solution in a cylindrical domain is obtained and the density of the elliptic potential is determined. Numerical results for the two-dimensional Helmholtz equation are also obtained.